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Phase-equivalent potentials obtained from supersymmetry

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Abstract. Given any central (or one-dimensional) potential, we discuss the general problem of using supersymmetry transformations to calculate families of potentials which are phase equivalent to it, but have 0, 1, 2, 3, ... less bound states. Various interesting properties of such families of potentials are derived, and it is shown how the previously studied Abraham-Moses and Pursey potentials emerge as limiting cases. For concreteness, the isospectral one-parameter families for two specific potentials of physical interest (the Coulomb and harmonic oscillator potentials) are fully worked out.

1. Introduction

Phase-equivalent potentials, that is potentials which have the same phase shifts and essentially the same bound-state spectrum have been discussed by many authors in a variety of contexts [1-7]. For example, in the context of $\alpha\alpha$ scattering, Baye [2] has recently discussed the ambiguities involved in determining the potential even when the phase shifts and bound states are precisely known. Seemingly unrelated potentials obtained by different authors from the same input phase shift data have in fact been shown to be related to supersymmetry (SUSY) transformations [2, 3]. In a similar spirit, Amado [4] has found potentials which are phase equivalent to the Coulomb potential, but which have one less bound state. Given their many physical applications, additional study and understanding of phase-equivalent potentials is clearly desirable. Here, we will review previous work and extend the discussion to any arbitrary central potential $V_-(r)$, $0 \leq r < \infty$. More specifically, we will show that (i) Amado's construction for the Coulomb potential is simply the previously developed Pursey procedure [5, 6] for constructing isospectral Hamiltonians with one less bound state but the same phase shifts; (ii) the procedure can be easily repeated to generate additional phase-equivalent Hamiltonians with 2, 3, ... less bound states; (iii) for any central potential $V(r)$ there exists a continuous one-parameter (λ) family of strictly isospectral Hamiltonians (i.e. exactly the same eigenvalue spectrum and S matrix as $V_-(r)$). When the parameter $\lambda \to 0$ (-1), one obtains the Pursey [5] (Abraham-Moses) [7] potentials, in which the ground state of $V_-(r)$ is absent. The supersymmetric partner potential $V_+(r)$ has the same bound-state spectrum as the Pursey and Abraham-Moses potentials. These results are developed in § 2 after a brief review of the formalism of supersymmetric quantum mechanics [8]. In § 3, the results are applied to the Coulomb and harmonic oscillator potentials. For illustrative purposes, we plot several members of the strictly isospectral family of potentials along with their ground-state wavefunctions. These plots give a

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nice intuitive feel for the manner in which a family is obtained. Some additional interesting properties of isospectral Hamiltonians are discussed in § 4. In particular, we show that successive repetition of the transformation which generates a one-parameter family of isospectral potentials does not yield more new families: the group theoretic structure of this transformation is clarified. It may be noted here that although we have so far talked about central potentials, exactly analogous results are also valid for one-dimensional potentials $V(x)$, $-\infty < x < \infty$.

2. Formalism

In supersymmetric quantum mechanics [8], the superpotential $W(x)$ determines the two-partner potentials

$$V_{\pm}(x) = W^2(x) \pm \mathrm{d}W/\mathrm{d}x \tag{2.1}$$

where the units have been chosen so that $\hbar = 2m = 1$. When supersymmetry is unbroken, the eigenstates of $V_{\pm}(x)$ are related by

$$E_{n+1}^{(-)} = E_n^{(+)} \qquad E_0^{(-)} = 0$$

$$\psi_n^{(+)} = \frac{A\psi_{n+1}^{(-)}}{[E_n^{(+)}]^{1/2}} \qquad \psi_{n+1}^{(-)} = \frac{A^+\psi_n^{(+)}}{[E_n^{(+)}]^{1/2}}, \tag{2.2}$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{2.3}$$

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}[\ln \psi_0^{(-)}(x)]. \tag{2.4}$$

The transmission and reflection amplitudes for the partner potentials $V_{\pm}(x)$ are related by [5, 9]

$$R_-(k) = \left(\frac{W_- + \mathrm{i}k}{W_- - \mathrm{i}k}\right) R_+(k)$$

$$T_-(k) = [(W_+ - \mathrm{i}k')/(W_- - \mathrm{i}k)]T_+(k) \tag{2.5}$$

where k and k' are given by

$$k = (E - W_-^2)^{1/2} \qquad k' = (E - W_+^2)^{1/2} \tag{2.6}$$

with

$$W_{\pm} = W(x = \pm\infty). \tag{2.7}$$

Similarly, for three-dimensional, central potentials the scattering matrices for the partner potentials are related by [9]

$$S_-(k') = \frac{(W_+ - \mathrm{i}k')}{(W_+ + \mathrm{i}k')} S_+(k'). \tag{2.8}$$

Note that for the three-dimensional case $W_+ = W(r = \infty)$ while $S(k') = \mathrm{e}^{\mathrm{i}\delta(k')}$ with $\delta(k')$ being the phase shift.

We would now like to raise the following interesting question: given the partner potential $V_+(x)$, is the potential $V_-(x)$ unique? In other words, for a given $V_+(x)$ what are the various forms of $W(x)$ and hence $V_-(x)$? Following Nieto [3], let us assume that there exists a more general superpotential $\hat{W}(x)$ which satisfies

$$V_+(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{2.9}$$

Clearly $\hat{W}(x) = W(x)$ is one of the solutions to (2.9). To find the most general solution let us set

$$\hat{W}(x) = W(x) + \phi(x) \tag{2.10}$$

in (2.9) and use (2.1). We then find that $y \equiv 1/\phi(x)$ satisfies the Bernoulli equation

$$y' = 1 + 2Wy \tag{2.11}$$

whose solution is

$$\frac{1}{y(x)} = \phi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln[I(x) + \lambda]. \tag{2.12}$$

The quantity $I(x)$ appearing in (2.12) is defined by

$$I(x) = \int_{-\infty}^{x} [\psi_0^{(-)}(x)]^2 \, \mathrm{d}x \tag{2.13}$$

where λ is a constant of integration and $\psi_0^{(-)}(x)$ is the normalised ground-state wavefunction of $V_-(x)$. Thus the most general $\hat{W}(x)$ satisfying (2.9) is given by

$$\hat{W}(x) = W(x) + \frac{\mathrm{d}}{\mathrm{d}x} \ln(I(x) + \lambda). \tag{2.14}$$

Consequently, there exists a one-parameter family of potentials $\hat{V}_-(x)$ given by

$$\hat{V}_-(x) = \hat{W}^2(x) - \hat{W}'(x) = V_-(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln[I(x) + \lambda] \tag{2.15}$$

all of whom have the same SUSY partner $V_+(x)$. Notice that $V_-(x)$ is itself a member of this one-parameter family since as $\lambda \to \pm\infty$, $\hat{V}(x) \to V_-(x)$.

From (2.4) and (2.14) it follows that the normalised ground-state wavefunction corresponding to the potential $\hat{V}_-(x)$ is

$$\hat{\psi}_0^{(-)}(x) = \frac{\sqrt{\lambda(1+\lambda)}\, \psi_0^{(-)}(x)}{[I(x) + \lambda]}. \tag{2.16}$$

This is an acceptable bound-state eigenfunction only if it is square integrable. Since $I(x)$ as given by (2.13) lies between 0 and 1, it is useful to distinguish between three different cases.

Case a. $\lambda > 0$ or $\lambda < -1$. In this case $\hat{\psi}_0^{(-)}(x)$ is an acceptable eigenfunction, since $\psi_0^{(-)}(x)$ is square integrable. Thus the bound-state spectrum of $\hat{V}_-(x)$ is identical to that of $V_-(x)$. Further, it follows from (2.13) and (2.14) that in this case $\hat{W}_\pm = W_\pm$ so that

$$\hat{R}_-(k) = R_-(k) \qquad \hat{T}_-(k) = T_-(k) \tag{2.17}$$

(and in the three-dimensional central potential case $\hat{S}_-(k') = S_-(k')$). Thus we see that for any potential $V_-(x)$ there is a one-parameter family of potentials given by (2.14) and (2.15) with $\lambda > 0$ or $\lambda < -1$) which are *strictly isospectral* to it, i.e. they have *same* bound state spectrum and same scattering matrix. Since $\lambda > 0$ or $\lambda < -1$, it is also clear that all members of the family have the same behaviour as $x \to \pm\infty$ (or $r \to 0, \infty$). It also follows from here that if the potential $V_-(x)$ is exactly solvable (i.e. E_n, ψ_n are all known) then $\hat{V}_-(x)$ is also exactly solvable with $E_n^{(-)} = \hat{E}_n^{(-)}$, $\hat{\psi}_0^{(-)}$ is given in terms of $\psi_0^{(-)}$ by (2.16) while $\hat{\psi}_{n+1}^{(-)}$ can be also easily obtained in terms of normalised eigenfunctions $\psi_{n+1}^{(-)}$. In particular, using (2.2) and (2.14) it easily follows that the normalised excited-state eigenfunctions for the potential $\hat{V}_-(x)$ are given by ($n = 0, 1, 2, \ldots$)

$$\hat{\psi}_{n+1}^{(-)}(x) = \psi_{n+1}^{(-)}(x) + \frac{1}{(E_{n+1}^{(-)})}\left(\frac{I'(x)}{I(x)+\lambda}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\psi_{n+1}^{(-)}(x). \tag{2.18}$$

This expression for the excited-state wavefunction is simpler than the one obtained by using inverse scattering theory (for a review of the inverse scattering approach, see [1]) where one is required to perform an indefinite integral like

$$\int_{-\infty}^{x} \psi_0^{(-)}(y)\psi_{n+1}^{(-)}(y)\,\mathrm{d}y.$$

Case b. $\lambda = 0$ or $\lambda = -1$. In either of these cases $I(x)+\lambda$ will vanish (at $x = -\infty$ or $x = +\infty$ respectively) so that $\hat{\psi}_0^{(-)}(x)$ is not square integrable. Thus, in both these cases SUSY is broken and hence the bound state spectrum is degenerate with that of $V_+(x)$. In other words, as $\lambda \to 0$ or -1, $\hat{V}_-(x)$ loses a bound state. For $\lambda = 0$, this was first pointed out by Pursey [5], and hence the corresponding potential will be termed $V^{(\mathrm{P})}(x)$. The situation for $\lambda = -1$ is equivalent to earlier work by Abraham and Moses [7] and hence we shall denote the corresponding potential by $V^{(\mathrm{AM})}(x)$. Let us discuss these cases separately.

(i) *Pursey potential* ($\lambda = 0$). The Pursey potential is obtained from $\hat{V}_-(x)$ by setting $\lambda = 0$, i.e.

$$V^{(\mathrm{P})}(x) = V_-(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln I(x) \tag{2.19}$$

while

$$W^{(\mathrm{P})}(x) = W(x) + \frac{\mathrm{d}}{\mathrm{d}x}\ln I(x). \tag{2.20}$$

Using (2.13), it immediately follows that

$$W_+^{(\mathrm{P})} = W_+ \qquad W_-^{(\mathrm{P})} = -W_-. \tag{2.21}$$

Since supersymmetry between $V_-(x)$ and $V_+(x)$ is unbroken, i.e. W_+ and W_- have opposite signs, it follows that $W_+^{(\mathrm{P})}$ and $W_-^{(\mathrm{P})}$ must have same signs. This implies that the supersymmetry between $V_+(x)$ and $V^{(\mathrm{P})}(x)$ is broken ($E_n^{(\mathrm{P})} = E_n^{(+)}$). In view of (2.5)-(2.8), one gets

$$R^{(\mathrm{P})}(k) = \left(\frac{W_- - \mathrm{i}k}{W_- + \mathrm{i}k}\right)^2 R_-(k) \tag{2.22}$$

$$T^{(\mathrm{P})}(k) = -\left(\frac{W_- - \mathrm{i}k}{W_- + \mathrm{i}k}\right) T_-(k) \tag{2.23}$$

while in the three-dimensional central potential case

$$S^{(P)}(k') = S_-(k'). \tag{2.24}$$

Thus, we have shown that for any central potential $V_-(r)$, the corresponding Pursey potential $V^{(P)}(r)$ as given by (2.19) is phase equivalent (or almost isospectral) to $V_-(r)$ (more generally to $\hat{V}_-(r)$) but has one less bound state. For some special choices of the potential this result has been obtained previously by Baye [2] and Amado [4]. The normalised eigenfunctions corresponding to $V^{(P)}(x)$ are readily obtained by setting $\lambda = 0$ in (2.18). One finds ($n = 0, 1, \ldots$)

$$\psi_n^{(P)}(x) = \psi_{n+1}^{(-)}(x) + \frac{1}{(E_{n+1}^{(-)})}\left(\frac{I'(x)}{I(x)}\right)\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^{(-)}(x). \tag{2.25}$$

Clearly, the procedure used to generate $V^{(P)}$ from V_- can be reapplied to $V^{(P)}$ with the help of (2.25) with $n = 0$. This new ground-state wavefunction $\psi_0^{(P)}$ will generate (for $\tilde{\lambda} = 0$) yet another isospectral potential with two bound states less than V_-, but the same phase shifts. It will also generate (for $-\infty < \tilde{\lambda} < -1$, $0 < \tilde{\lambda} < \infty$) a new family of potentials which are strictly isospectral to $V^{(P)}$.

(ii) *Abraham-Moses potential* ($\lambda = -1$). The Abraham-Moses potential is obtained from $\hat{V}_-(x)$ by setting $\lambda = -1$, i.e.

$$V^{(AM)}(x) = V_-(x) - 2\frac{d^2}{dx^2}\ln(I(x) - 1) \tag{2.26}$$

while

$$W^{(AM)}(x) = W(x) + \frac{d}{dx}\ln(I(x) - 1). \tag{2.27}$$

From (2.13) it then follows that

$$W_+^{(AM)} = -W_+ \qquad W_-^{(AM)} = W_- \tag{2.28}$$

i.e. the supersymmetry between $V_+(x)$ and $V^{(AM)}(x)$ is broken ($E_n^{(AM)} = E_n^{(+)}$). From (2.5)-(2.8) it also follows that

$$R^{(AM)}(k) = R_-(k)$$

$$T^{(AM)}(k) = -\left(\frac{W_+ + ik'}{W_+ - ik'}\right)T_-(k) \tag{2.29}$$

while in the three-dimensional central case

$$S^{(AM)}(k') = \left(\frac{W_+ + ik'}{W_+ - ik'}\right)^2 S_-(k'). \tag{2.30}$$

The normalised eigenfunctions corresponding to $V_-^{(AM)}(x)$ can be shown to be ($n = 0, 1, 2, \ldots$)

$$\psi_n^{(AM)}(x) = \psi_{n+1}^{(-)}(x) + \frac{1}{(E_{n+1}^{(-)})}\frac{I'(x)}{(I(x) - 1)}\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^{(-)}(x). \tag{2.31}$$

Case c. $-1 < \lambda < 0$. In this case the potential $\hat{V}_-(x)$ given by (2.15) is singular at some finite value of x. We shall reject such singular potentials on physical grounds.

3. Isospectral potentials to the Coulomb and oscillator potentials

To elucidate the discussion of the last section, it may be worthwhile to present a few examples. With this in mind, we shall now explicitly obtain the one-parameter family of isospectral potentials corresponding to the Coulomb and three-dimensional harmonic oscillator potentials.

3.1. Coulomb potential

As is well known [8], in this case the superpotential is

$$W(r) = -\frac{(l+1)}{r} + \frac{e^2}{2(l+1)} \tag{3.1}$$

so that

$$V_-(r) = \frac{e^4}{4(l+1)^2} + \frac{l(l+1)}{r^2} - \frac{e^2}{r} \tag{3.2}$$

while

$$V_+(r) = \frac{e^4}{4(l+1)^2} + \frac{(l+1)(l+2)}{r^2} - \frac{e^2}{r}. \tag{3.3}$$

Using (2.4) the normalised ground-state eigenfunction corresponding to the potential $V_-(r)$ can be shown to be

$$\psi_0^{(-)}(r) = \left(\frac{e^2}{l+1}\right)^{l+3/2} \frac{r^{l+1}}{[(2l+2)!]^{1/2}} \exp[-e^2 r/2(l+1)]. \tag{3.4}$$

Now it is straightforward to compute $I(r)$ as defined by (2.13). We find

$$I(r) = \gamma\left(2l+3; \frac{e^2 r}{(l+1)}\right)[(2l+2)!]^{-1} \tag{3.5}$$

where

$$\gamma(a; b) \equiv \int_0^b y^{a-1} e^{-y}\, dy$$

is an incomplete gamma function. Using the formalism of the last section and the normalised Coulomb wavefunctions it is then straightforward to compute $\hat{V}_-(r)$, $V^{(P)}(r)$, $V^{(AM)}(r)$ and the corresponding wavefunctions. Notice that $\hat{V}_-(r)$ is strongly l dependent. In figures 1-4, we have plotted some of them for the special case of $l = 0$ and have chosen $e^2 = 2$. In figure 1, we see that as λ varies from $+\infty$ to 0 (i.e. as one moves away from the Coulomb potential), the (strictly isospectral) potentials start developing a maximum. As λ becomes smaller and smaller, the value of the maximum goes on increasing and the corresponding position starts shifting towards $r = 0$. Finally as λ becomes zero, this maximum shifts to $r = 0$ so that there is a net gain of two units of angular momentum, but in this process one bound state gets lost. This is the limit in which one goes over to the Pursey potential $V^{(P)}(r)$. It is quite noticeable from the figure that the curves for $\lambda = 0.02$ and $\lambda = 0$ hug each other over a wide region of r. In figure 2, we have plotted the family of potentials as λ varies from $-\infty$ to -1 with

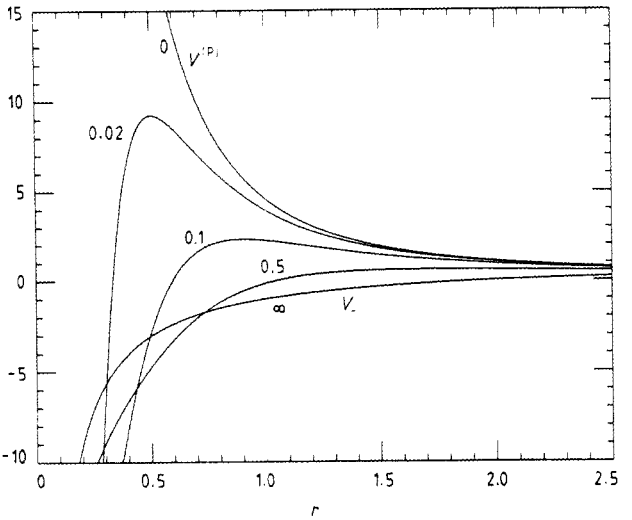

Figure 1. Potentials with $0 < \lambda < \infty$ which are strictly isospectral to the Coulomb potential $V_-(r)$ given by (3.2) with $l = 0$, $e^2 = 2$. When $\lambda = 0$, one has the Pursey potential $V^{(P)}(r)$, which has one less bound state. $V^{(P)}(r)$ and $V_-(r)$ have the same phase shifts.

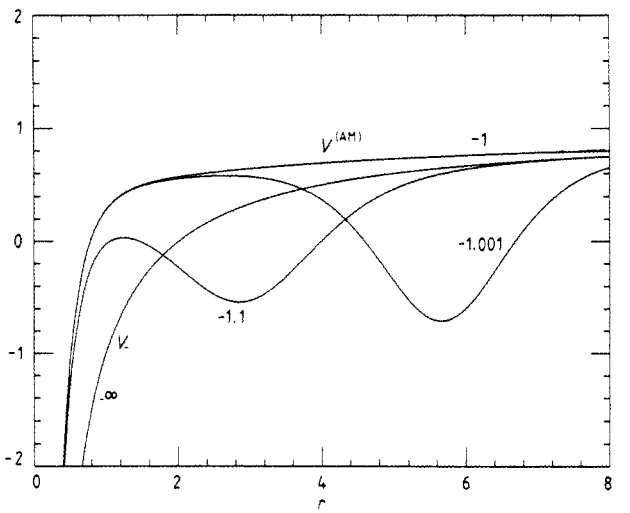

Figure 2. Potentials with $-\infty < \lambda < -1$ which are strictly isospectral to the Coulomb potential $V_-(r)$ of figure 1. When $\lambda = -1$, one has the Abraham-Moses potential $V^{(AM)}(r)$, which has one less bound state. $V^{(AM)}$ and $V_-(r)$ do *not* have the same phase shifts.

the $\lambda = -\infty$ potential being the original Coulomb potential. Here we note that as λ approaches -1, the potential develops a minimum which shifts towards $r = \infty$ as λ is varied. Finally, as λ becomes -1, this attractive potential well shifts to $r = \infty$ and is lost, thereby losing one bound state. This is the limit in which one goes over to the Abraham-Moses potential $V^{(AM)}(r)$. In figure 3, we have plotted the three potentials $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ which have identical bound state spectra (but which are not isospectral as they have different phase shifts). In particular, from (2.8),

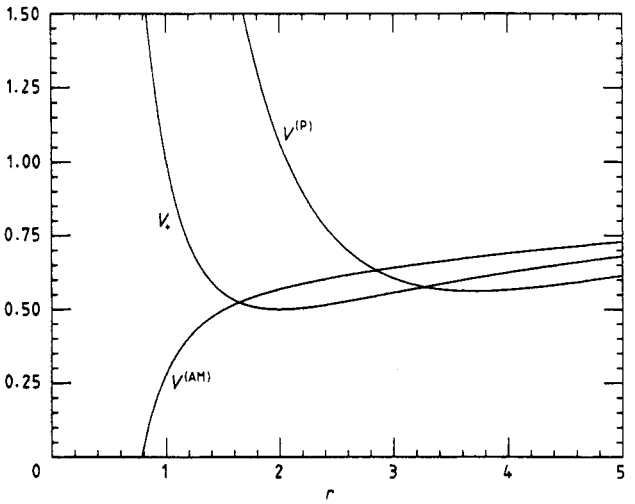

Figure 3. The Pursey potential $V^{(P)}(r)$, Abraham-Moses potential $V^{(AM)}(r)$ and SUSY partner potential $V_+(r)$ for the Coulomb potential $V_-(r)$ of figure 1. All these potentials have the same bound-state spectrum, but different phase shifts.

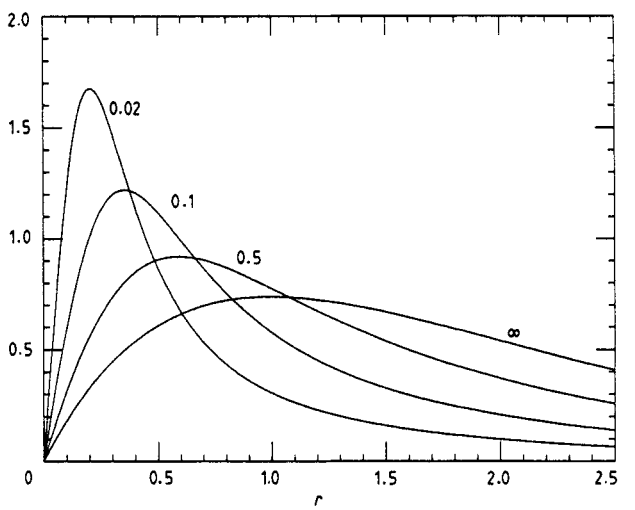

Figure 4. Ground-state wavefunctions $\hat{\psi}_0(r)$ for all the potentials drawn in figure 1, except $V^{(P)}(r)$.

(2.29) and (2.30) it follows that

$$S_+(k') = \left(\frac{e^2/2(l+1)+ik'}{e^2/2(l+1)-ik'}\right) S_-(k')$$

$$S^{(P)}(k') = S_-(k')$$

$$S^{(AM)}(k') = \left(\frac{e^2/2(l+1)+ik'}{e^2/2(l+1)-ik'}\right)^2 S_-(k') \tag{3.6}$$

where

$$S_{-}(k')=\frac{\Gamma[e^2/2(l+1)+ik']\Gamma[e^2/2-ik']}{\Gamma[e^2/2(l+1)-ik']\Gamma[e^2/2+ik']}. \tag{3.7}$$

From the figure it is clear that the usual way of obtaining the partner potential with one bound state less (Darboux procedure) is sort of intermediate between the Pursey and Abraham-Moses procedures. Whereas in the Pursey approach, one 'lifts' the potential up from the left, in the Abraham-Moses approach one 'lifts' it up from the

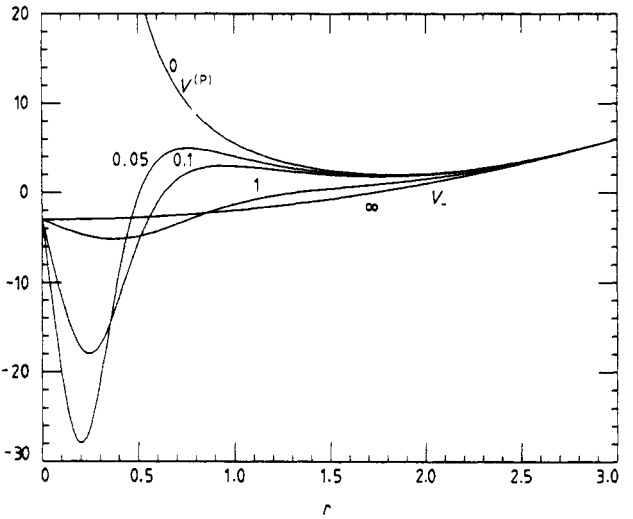

Figure 5. Various potentials with $0<\lambda<\infty$ which are strictly isospectral to the harmonic oscillator potential $V_{-}(r)$ given by (3.7) with $l=0$, $\omega=2$. When $\lambda=0$, one has the Pursey potential $V^{(P)}(r)$ with one less bound state.

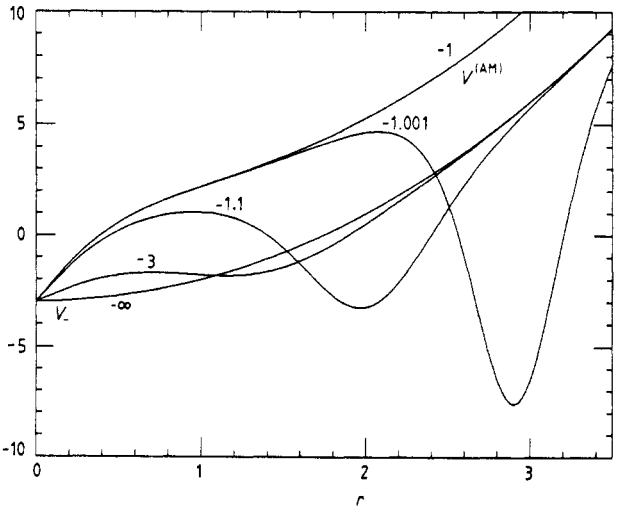

Figure 6. Potentials with $-\infty<\lambda<1$ which are strictly isospectral to the harmonic oscillator potential $V_{-}(r)$ of figure 5. The Abraham-Moses potential $V^{(AM)}(r)$ has one less bound state and corresponds to $\lambda=-1$.

right. In the Darboux approach one 'lifts' from both ends, but by a smaller amount than in the Pursey or AM construction. We shall in fact rigorously show in the next section that if for $r \to 0$ we have $V_-(r) \sim l(l+1)/r^2$, then $V_+(r) \sim (l+1)(l+2)/r^2$, $V^{(P)}(r) \sim (l+2)(l+3)/r^2$ while $V^{(AM)}(r) \sim l(l+1)/r^2$. Note that the phase equivalence of the Coulomb potential and the corresponding Pursey potential (equation (3.6)) was pointed out by Amado [4].

In figure 4 we have plotted the ground-state wavefunctions for the isospectral family $\hat{V}_-(r)$.

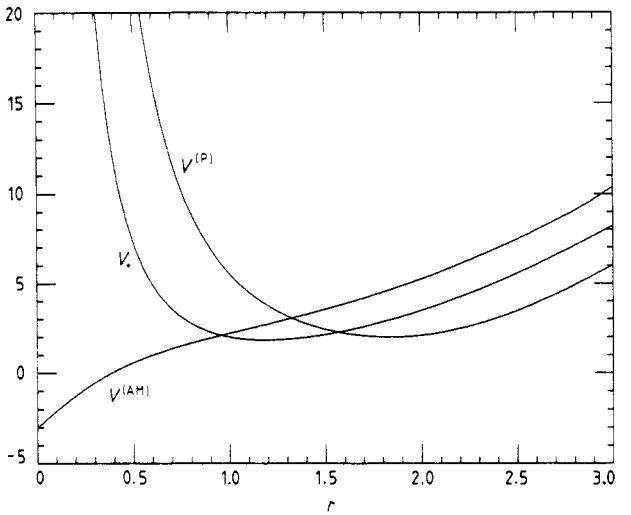

Figure 7. The Pursey potential $V^{(P)}(r)$, Abraham-Moses potential $V^{(AM)}(r)$ and SUSY partner potential $V_+(r)$ for the three-dimensional harmonic oscillator potential $V_-(r)$ of figure 5. All these potentials have identical bound-state spectra.

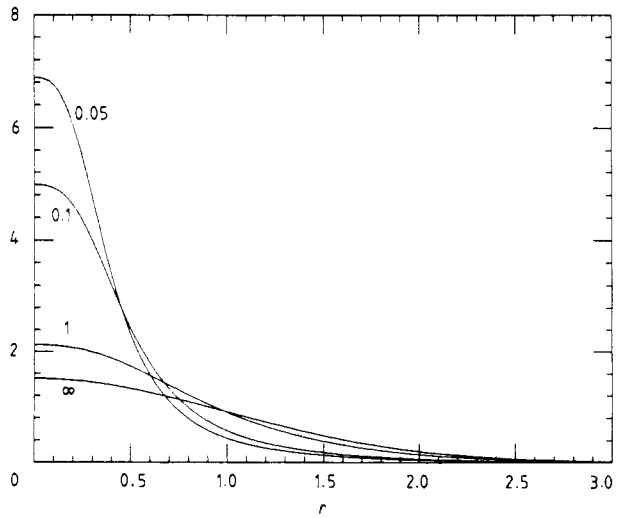

Figure 8. Ground-state wavefunctions $\hat{\psi}_0(r)/r$ for all the potentials drawn in figure 5, except $V^{(P)}(r)$.

3.2. Harmonic oscillator potential.

Proceeding analogously we have in this case

$$W(r)=\frac{\omega}{2}r-\frac{(l+1)}{r}$$

$$V_-(r)=\frac{\omega^2}{4}r^2+\frac{l(l+1)}{r^2}-\omega(l+3/2)$$

$$V_+(r)=\frac{\omega^2}{4}r^2+\frac{(l+1)(l+2)}{r^2}-\omega(l+1/2) \tag{3.8}$$

$$\psi_0^{(-)}(r)=\frac{\omega^{(2l+3)/4}}{2^{(2l+1)/4}}\frac{r^{l+1}}{[\Gamma(l+3/2)]^{1/2}}\exp(-\omega r^2/4). \tag{3.9}$$

Hence,

$$I(r)=\gamma(l+3/2;\,\omega r^2/2)/\Gamma(l+3/2). \tag{3.10}$$

From here one can readily compute $\hat{V}_-(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ and the corresponding energy eigenfunctions. For the special case of $l=0$ (and choosing $\omega=2$) these have been plotted in figures 5–8. It may be noted here that in this confining potential case $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ are strictly isospectral. However, they do not belong to a single one-parameter family as they behave very differently for $r\to 0$. In this case, one finds that as $|\lambda|$ decreases from ∞ to 0 the potential develops a minimum which becomes deeper and narrower as $|\lambda|$ decreases. As $\lambda\to 0$ this attractive deep potential function moves towards $r=0$ and is eventually lost, thereby taking away one bound state.

4. Some properties of isospectral Hamiltonians

The formalism developed in § 2 as well as the discussion of the last section raises several questions. Some of them are the following.

(i) By starting with a given potential $V_-(r)$, one can generate a one-parameter family of potentials $\hat{V}_-(r)$. An obvious question is that if one now starts with any particular member $\hat{V}_-(r,\lambda)$ and again constructs a one-parameter family of isospectral Hamiltonians then does one generate another distinct family? If yes, then in this way one would have an infinite number of one-parameter families of isospectral Hamiltonians! We show below that this is not the case.

(ii) We saw in § 2 that for any potential $V_-(r)$, one can always construct the phase-equivalent Pursey potential $V^{(P)}(r)$ having one bound state less. An obvious question is whether one could repeat this construction and obtain a series of phase-equivalent potentials with 2, 3, 4, ... bound states less than $V_-(r)$.

(iii) We have seen in § 2 that even though $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ have the same bound-state spectrum, in general they have different scattering matrices. The question is in what way do these potentials differ from each other?

(iv) Just as there are three isospectral families, i.e. one for $V_+(r)$, one for $V^{(P)}(r)$ and one for $V^{(AM)}(r)$, what about the three analogous families having the same bound-state spectrum as $V_-(r)$? How does one find these families and do they always exist?

We will now answer and discuss the four questions raised above.

4.1. Uniqueness of the one-parameter family

Here we wish to show that the one-parameter family $\hat{V}_-(x)$ as given by (2.15) is unique, i.e. if one starts with $\hat{V}_-(x, \lambda)$ and works out its one-parameter family $\hat{\hat{V}}_-(x)$ then one does not get anything new. The proof is rather straightforward.

Let us start with the one-parameter family of potential $\hat{V}_-(x)$ as given by (2.15) for a given value of λ outside the interval $[-1, 0]$. Clearly its one-parameter family of potentials would be given by

$$\hat{\hat{V}}_-(x) = \hat{V}_-(x) - 2\frac{d^2}{dx^2}\ln(\hat{I}(x)+\hat{\lambda}) = V_-(x) - 2\frac{d^2}{dx^2}\ln[(I+\lambda)(\hat{I}+\hat{\lambda})] \tag{4.1}$$

where (see (2.16))

$$\hat{I}(x) = \int_{-\infty}^{x} dy\hat{\psi}_0^2(y) = \lambda(1+\lambda)\int_{-\infty}^{x} dy\frac{\psi_0^2(y)}{(I(y)+\lambda)^2}. \tag{4.2}$$

Since $\psi_0^2(y) = I'(y)$, the integral can be evaluated to give

$$\hat{I}(x) = (1+\lambda) - \frac{\lambda(1+\lambda)}{I(x)+\lambda}. \tag{4.3}$$

Therefore

$$\begin{aligned}\hat{\hat{V}}_-(x) &= V_-(x) - 2\frac{d^2}{dx^2}\ln[(1+\lambda+\hat{\lambda})(I(x)+\lambda) - \lambda(1+\lambda)] \\ &= V_-(x) - 2\frac{d^2}{dx^2}\ln[I(x)+q] \qquad q = \frac{\lambda\hat{\lambda}}{1+\lambda+\hat{\lambda}}.\end{aligned} \tag{4.4}$$

Thus the second iteration does not give anything new as $\hat{\hat{V}}_-(x)$ has a similar form to $\hat{V}_-(x)$ except that λ has been changed to q as given by (4.4). Note that the parameter q remains in the range $-\infty < q < -1$, $0 < q < \infty$, whenever both λ and $\hat{\lambda}$ are in the same ranges. In fact, it is easy to show that the set of transformations which generate one-parameter families form a group, with the law of composition given by (4.4).

4.2. The number of almost isospectral Hamiltonians

In § 2 we showed that for any central potential $V_-(r)$, one can always construct the corresponding Pursey potential $V^{(P)}(r)$ which is phase equivalent to $V_-(r)$ but has one bound state less. Clearly one can repeat this construction further and treating $V^{(P)}(r)$ as the starting potential one can calculate its corresponding Pursey potential $V^{(PP)}(r)$, which then clearly will be phase equivalent to $V_-(r)$ but will have two bound states less. In this way if $V_-(r)$ has n bound states, then one can construct $n+1$ almost isospectral (i.e. phase-equivalent) Hamiltonians with $0, 1, 2, \ldots, n$ bound states less. Two remarks are in order at this stage. Firstly, clearly for all confining potentials with pure bound-state spectrum, $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ and their families are strictly isospectral. Further, all of them are almost isospectral to $V_-(r)$ with one bound state less and so on. Secondly, for symmetric reflectionless potentials in one dimension, $T_+(k) = T^{(P)}(k) = T^{(AM)}(k)$ (and of course $R_+ = R^{(P)} = R^{(AM)} = 0$) so that they are also strictly isospectral and all of them again are almost isospectral to $V_-(x)$ with one less bound state.

4.3. The $r \to 0$ behaviour of various isospectral Hamiltonians

Here we want to show that if for small r ($r \to 0$) any central potential $V_-(r) \sim l(l+1)/r^2$ then $V_+(r) \sim (l+1)(l+2)/r^2$, $V^{(P)}(r) \sim (l+2)(l+3)/r^2$ and $V^{(AM)}(r) = l(l+1)/r^2$ [10]†. The proof is fairly straightforward. First, if $V_-(r) \sim l(l+1)/r^2$ as $r \to 0$, then the superpotential $W(r)$ as given by (2.4) behaves like (since $\psi_0(r) \sim r^{l+1}$)

$$W(r) \sim -\frac{(l+1)}{r} \tag{4.5}$$

and hence

$$V_+(r) = W^2(r) + W'(r) \sim \frac{(l+1)(l+2)}{r^2}. \tag{4.6}$$

However, for a given $V_-(r)$, the most general $\hat{W}(r)$ and $\hat{V}_-(r)$ are as given by (2.14) and (2.15). Since $0 \leq I(r) \leq 1$ it is clear that for any non-zero λ (including $\lambda = -1$, i.e. the Abraham-Moses potential),

$$\hat{W}(r) \sim \frac{(l+1)}{r} \qquad \hat{V}(r) \sim \frac{l(l+1)}{r^2}. \tag{4.7}$$

In the special case of $\lambda = 0$, i.e. the Pursey potential, this is no longer true. Since $\psi_0(r) \sim Ar^{l+1}$, as $r \to 0$, it follows that

$$I(r) \sim A^2 r^{2l+3}/(2l+3) \tag{4.8}$$

so that

$$\frac{I'(r)}{I(r)} \sim \frac{(2l+3)}{r}. \tag{4.9}$$

Hence, as $r \to 0$,

$$W^{(P)}(r) \sim \frac{(l+2)}{r} \qquad V^{(P)}(r) \sim \frac{(l+2)(l+3)}{r^2}. \tag{4.10}$$

It may be noted here that even though $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ behave differently for $r \to 0$, all of them have identical leading asymptotic behaviour as $r \to \infty$.

4.4. Addition of a bound state below the ground state

We have seen that starting with a potential $V_-(r)$, one can obtain three distinct families of potentials $V_+(r)$, $V^{(P)}(r)$ and $V^{(AM)}(r)$ with the same bound-state spectra. On the other hand, corresponding to $V_-(r)$, we have only been able to construct one family of isospectral potentials, i.e. $\hat{V}_-(r)$. What about the other two families of potentials? How can one construct these? Clearly, the recipe would be to treat $V_-(r) \equiv U_+(r)$ and obtain the corresponding partner potential $U_-(r)$ (with one additional bound state) and hence $\hat{U}_-(r)$. In practice the way to do this is to find the solution $\phi(r)$ of the Schrödinger equation for potential $V_-(r)$ with zero eigenvalue, such that $\phi(r)$ diverges both at $r \to 0$ and $r \to \infty$, so that $1/\phi(r)$ will be well behaved in both limits [5]. Clearly this procedure can be done until we arrive at the potential with zero centrifugal barrier,

† For the supersymmetric partner potential $V_+(r)$, this was previously noticed by Imbo and Sukhatme in [10].

i.e. for which $\psi_0(r) \sim r$ as $r \to 0$. The obvious point which we would like to make here is that one cannot add any more states below it, since then the new potential $U_-(r)$ would have a centrifugal barrier of -1 and correspond to a divergent wavefunction at $r = 0$, which is unacceptable.

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